M11/5/MATHL/HP1/ENG/TZ2/XX

22117205

MATHEMATICS
HIGHER LEVEL
PAPER 1

Wednesday 4 May 2011 (afternoon)

2 hours

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

2211-7205

0113

13 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 6]*

The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

(a) Find the value of p and the value of q. *[4 marks]*

(b) The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph. *[2 marks]*

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2. *[Maximum mark: 5]*

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ a & -1 \end{pmatrix}$.

(a) Find the matrix $\boldsymbol{A}^2$. *[2 marks]*

(b) If $\det \boldsymbol{A}^2 = 16$, determine the possible values of a. *[3 marks]*

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3. *[Maximum mark: 6]*

The random variable X has probability density function f where

$$f(x) = \begin{cases} kx(x+1)(2-x), & 0 \le x \le 2 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Sketch the graph of the function. You are not required to find the coordinates of the maximum. *[1 mark]*

(b) Find the value of k. *[5 marks]*

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4. *[Maximum mark: 6]*

The complex numbers $z_1 = 2 - 2\text{i}$ and $z_2 = 1 - \sqrt{3}\text{i}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

(a) find AB, giving your answer in the form $a\sqrt{b - \sqrt{3}}$, where $a, b \in \mathbb{Z}^+$; *[3 marks]*

(b) calculate $\text{A}\hat{\text{O}}\text{B}$ in terms of π. *[3 marks]*

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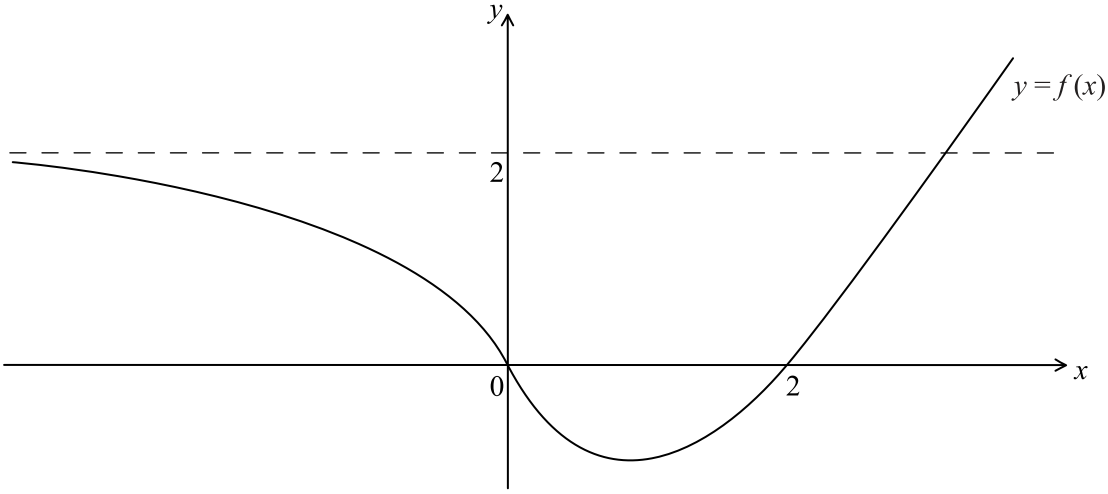
5. *[Maximum mark: 6]*

The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 2$.

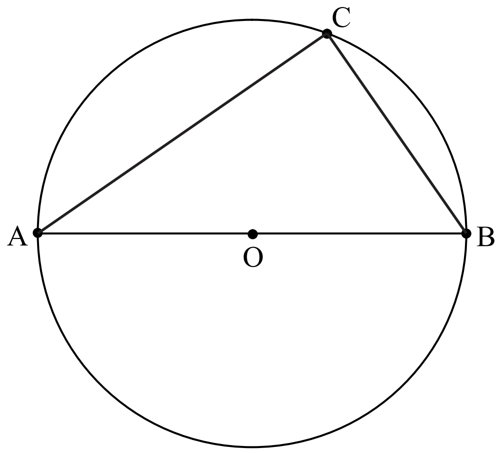


(a) Sketch the graph of $y = \dfrac{1}{f(x)}$. *[3 marks]*

(b) Sketch the graph of $y = x\,f(x)$. *[3 marks]*

6. *[Maximum mark: 5]*

In the diagram below, [AB] is a diameter of the circle with centre O. Point C is on the circumference of the circle. Let $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.



(a) Find an expression for $\overrightarrow{CB}$ and for $\overrightarrow{AC}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$. *[2 marks]*

(b) Hence prove that $\hat{ACB}$ is a right angle. *[3 marks]*

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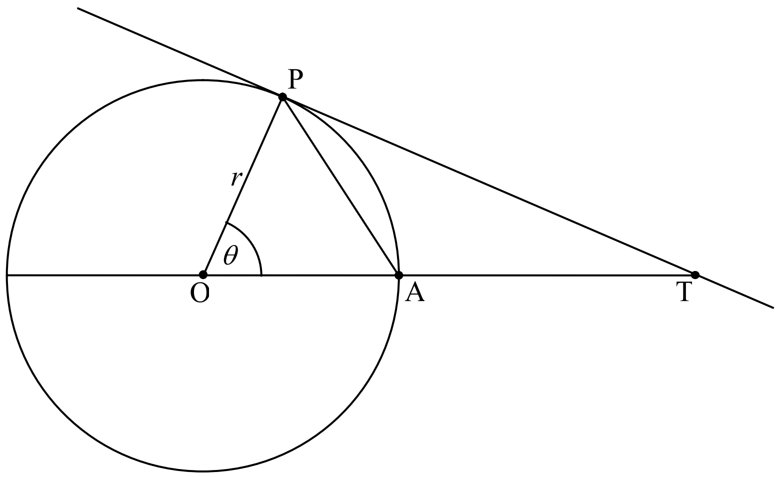
7. *[Maximum mark: 5]*

The diagram shows a tangent, (TP), to the circle with centre O and radius r. The size of $\text{P}\hat{\text{O}}\text{A}$ is θ radians.



(a) Find the area of triangle AOP in terms of r and θ. *[1 mark]*

(b) Find the area of triangle POT in terms of r and θ. *[2 marks]*

(c) Using your results from part (a) and part (b), show that $\sin\theta < \theta < \tan\theta$. *[2 marks]*

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8. *[Maximum mark: 6]*

A function is defined by $h(x) = 2\mathrm{e}^x - \frac{1}{\mathrm{e}^x}$, $x \in \mathbb{R}$. Find an expression for $h^{-1}(x)$.

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9. *[Maximum mark: 7]*

A batch of 15 DVD players contains 4 that are defective. The DVD players are selected at random, one by one, and examined. The ones that are checked are not replaced.

(a) What is the probability that there are exactly 3 defective DVD players in the first 8 DVD players examined? *[4 marks]*

(b) What is the probability that the 9^{th} DVD player examined is the 4^{th} defective one found? *[3 marks]*

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10. *[Maximum mark: 8]*

An arithmetic sequence has first term a and common difference d, $d \neq 0$. The 3rd, 4th and 7th terms of the arithmetic sequence are the first three terms of a geometric sequence.

(a) Show that $a = -\frac{3}{2}d$. *[3 marks]*

(b) Show that the 4th term of the geometric sequence is the 16th term of the arithmetic sequence. *[5 marks]*

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*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 15]*

The curve C has equation $y = \frac{1}{8}(9 + 8x^2 - x^4)$.

(a) Find the coordinates of the points on C at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[4 marks]*

(b) The tangent to C at the point $\mathrm{P}(1, 2)$ cuts the x-axis at the point T. Determine the coordinates of T. *[4 marks]*

(c) The normal to C at the point P cuts the y-axis at the point N. Find the area of triangle PTN. *[7 marks]*

12. *[Maximum mark: 20]*

(a) Factorize $z^3 + 1$ into a linear and quadratic factor. *[2 marks]*

Let $\gamma = \frac{1 + \mathrm{i}\sqrt{3}}{2}$.

(b) (i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$. *[9 marks]*

The matrix $\boldsymbol{A}$ is defined by $\boldsymbol{A} = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}$.

(c) Show that $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \boldsymbol{0}$, where $\boldsymbol{0}$ is the zero matrix. *[4 marks]*

(d) Deduce that

(i) $\boldsymbol{A}^3 = -\boldsymbol{I}$;

(ii) $\boldsymbol{A}^{-1} = \boldsymbol{I} - \boldsymbol{A}$. *[5 marks]*

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

13. *[Maximum mark: 25]*

(a) (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \le x \le \frac{\pi}{2}$.

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \le x \le \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs. *[9 marks]*

(b) Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$. *[8 marks]*

(c) The increasing function f satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$.

(i) By reference to a sketch, show that $\int_0^a f(x)\,\mathrm{d}x = ab - \int_0^b f^{-1}(x)\,\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$. *[8 marks]*